University of Baghdad
College of Sciences for Women
Mathematics Department
Third Class
Semester two

Modules theory

By

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In this semester, we shall study the following four chapters:

Chapter one: Definitions and Preliminaries.

Chapter two: Modules homomorphism

Chapter three: Sequences

Chapter four: Noetherian and Artinian modules

Chapter one (Definitions and Preliminaries.)

Definition. (***Modules***) Let R be a ring. A (left) R-module is an additive abelian group M together with a function $f : R \times M \to M$ defined by: $f(r,a)=ra$ such that for all $r,s \in R$ and $a,b \in M$:

1. $r(a + b) = ra + rb$.
   }(distributive laws)
2. $(r + s)a = ra + sa$.
3. $r(sa) = (rs)a$. (associative law)

lf R has an identity element 1_R and

4. $1_R a = a$ for all $a \in M$,

then M is said to be a ***unitary*** left R-module.

Remarks.

1. A (unitary) right R-module is defined similarly by a function $f: M x R \to M$ denoted by $(a,r) \to ar$ and satisfying the obvious analogues of (1)-(4).

2. If R is commutative, then every left R-module M can be given the structure of a right R-module by defining $ar = ra$ for $r \in R$, $a \in M$.

3. Every module M over a commutative ring R is assumed to be both a left and a right module with $ar = ra$ for all $r \in R$, $a \in M$.
4. We shall refer to left R-module by R- module. Also, in this course, all R-modules are unitary.

Remarks.

1. If 0_M is the additive identity element of M and 0_R is the additive identity element of a ring R (where M is an R-module), then for all $r \in R$, $a \in M$: $r\, 0_M = 0_M$ and $0_R.\, a = 0_M$.
2. $(-r)a = -(ra) = r(-a)$ and $n(ra) = r(na)$ for all $r \in R$, $a \in M$ and $n \in \mathbb{Z}$(ring of integers).

Examples.

1. Every commutative ring is an R-module.

   Proof. Define $f: R \times R \to R$ by $f(r_1, r_2) = r_1r_2$ for all $r_1, r_2 \in R$.then
   a. $(r_1+r_2)r = r_1r + r_2r$
   b. $r(r_1+ r_2) = rr_1+ rr_2$
   c. $(r_1r_2)r = r_1(r_2r)$

2. Every additive abelian group G is a unitary $\mathbb{Z}$ -module.

   Proof. Define $\alpha: \mathbb{Z} \times G \to G$ by: $\alpha(n, m) = nm$ for all $n \in \mathbb{Z}$ and $m \in G$.

   i.e $\alpha(n, m) = \underbrace{m + m + \cdots + m}_{n-times} = nm$

   since G is group and $m \in G$, then there is $-m \in G$ such that

$(-nm) = \underbrace{-m - m - \cdots - m}_{n-times}$

Now,

i. $(n_1+n_2)m = n_1m + n_2m$

ii. $n(m_1+ m_2) = \underbrace{(m_1 + m_2) + (m_1 + m_2) + \cdots + (m_1 + m_2)}_{n-times}$

$= nm_2 + nm_2$

iii. $(n_1 n_2)m = n_1(n_2m)$

also, since $\mathbb{Z}$ has identity element, then

iv. $1. m = m$

3. Every ideal in a ring R is an R- module
4. Every vector space V over a field F is F-module.
5. If Q is the set of rational numbers, then Q is $\mathbb{Z}$-module.
   Proof. Define $\beta: \mathbb{Z} \text{ x } Q \rightarrow Q$ by:
   $\beta(m,\frac{n}{t}) = m\frac{n}{t} = \frac{mn}{t}$ for all $m\in \mathbb{Z}$ and $\frac{n}{t} \in Q$.
6. If $\mathbb{Z}_n$ is the group of integers modulo n, then $\mathbb{Z}_n$ is $\mathbb{Z}$-module.
   Proof. define $\alpha: \mathbb{Z} \text{ x } \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ by: $\alpha(n, \bar{a}) = n\bar{a}$ for all $n\in \mathbb{Z}$, $\bar{a} \in \mathbb{Z}_n$.
7. Let A be an abelian group and
   $S= end_R(A) = Hom_R(A, A)= \{f: A\rightarrow A;$ f is a group homomorphism$\}$
   Define " + " on S by: for all f, g ∈ S and a ∈ A,
   $(f+g)(a) = f(a) + g(a)$
   Then
   1. (S, +) is an abelian group:
      i. S is closed under "+"
      ii. $0(a) = 0$ (zero function $0 : A\rightarrow A$)
      iii. $(-f(a)) = -(f(a))$ (additive inverse)

      $(f+ (-f)(a) = f(a) + -(f(a)) = 0$

      iv. "+" is an associative operation

      iv. "+" is an abelian:

$(f+g)(a) = f(a) + g(a) = g(a) + f(a) = (g+f)(a)$

(S, +) is an abelian group

2. Define " . " on S by: for all $f, g \in S$ and $a \in A$,
   $f.g \equiv fog$ and $(fog)(a) = (f(g(a))$
   (S, +, .) is a ring with identity $f = I: A \rightarrow A$ (where $foI = Iof = f$)
3. Now, one can consider A as a unitary S-module:
   with $\alpha : S \times A \rightarrow A$, $\alpha(f, a) = f(a)$ $f \in S$ and $a \in A$

8. If R is a ring, every abelian group can be consider as an R-module with trivial module structure by defining ra =0 for all $r \in R$ and $a \in A$.
9. ***The R-module $M_{n.}(R)$***. let

$$M_{n.}(R) = \text{the set of nxn matrices over } R$$
$$=\{(a_{ij})_{nxn} \mid a \in R\}$$

$M_n.(R)$ is an additive abelian group under matrix addition. If $(a_{ij})\in M_{n.}(R)$ and $a\in R$, then the operation $a.(a_{ij}) = (a.a_{ij})$ makes $M_{n.}(R)$ into an R-module. $M_{n.}(R)$is also a left R-module under the operation $a.(a_{ij})= (a.a_{ij})$.

10.***The Module R[X]***. If R[X] is the set of all polynomials in X with their coefficients in R,
i.e $R[X] = \{(a_0 ,a_1 ,\dots ,a_n)\mid a_i\in R, i = 1,2,\dots,n,\}$
then (R[X], +) is an additive abelian group under polynomial addition on R[X] is an R-module via the function $R \times R[X]\rightarrow R[X]$ defined by : $a.(a_0 + X.a_1 +\dots + X^n.a_n) = (a.a_0) +(a.a_1).X + \dots + (a.a_n).X^n$

<u>Definition.</u> Let R be a ring, A an R-module and B a nonempty subset of A. B is a ***submodule*** of A provided that B is an additive subgroup of A and $rb \in B$ for all $r \in R$ and $b \in B$.

Remark. Let R be a ring, A an R-module and B a nonempty subset of A. B is a submodule iff:
1. for all $a, b \in B$, $a+b \in B$
2. for all $r \in R$ and $a \in B$, $ra \in B$.

Another characterization for a submodule concept

Remark. A nonempty subset B of an R-module A a submodule iff: $ax + by \in B$, for all $a, b \in R$ and $x, y \in B$.

Examples.

1. let M an R-module and $x \in M$, the set

$$R_x = \{rx \mid r \in R\} \text{ is a submodule of M such that}$$

a. $r_1x - r_2x = r_1x + (-r_2)x \in R_x$.

b. $r_1(r_2x) = (r_1r_2)x$

2. let R be a commutative ring with identity and S be a set. Consider the set

$$X = R^s = \{f : S \to R;\ f \text{ is a function}\}.$$

The two operation "+" and "." on X denoted by

$(f+g)(s) = f(s) + g(s)$ and $(f.g)(s) = f(s) . g(s)$ for $s \in S$ and $f, g \in X$

Then (X, +) is an abelian group (H.W)

The function $\alpha : R \times X \to X$ denoted by $\alpha(r, f) = rf$ since $(rf)(s)= r(f(s))$ for all $s \in S$, $r \in R$ and $f \in X$, then X is an R-module(H. W)

And $Y = \{f : \in X : f(s) = 0$ for all but at most a finite number of $s \in S\}$, the Y is a submodule of an R-module X. (H.W)

3. ***Finite Sums of Submodules.*** If $M_1, M_2, \ldots, M_n$ are submodules of an R-module M, then $M_1+ M_2+ \ldots +M_n = \{x_1+ x_2+ \ldots +x_n \mid x_i \in M_i$ for $i = 1,2,\ldots,n\}$ is a submodule of M for each integer $n \geq 1$.

4. If one take n=2 in (3) then

$$N+K=\{x+y \mid x \in N, y \in K\}$$

is a submodule of M for each submodule N and K of M

Proof. let w_1 , $w_2 \in N+K$. Then

i. $w_1= x_1 + y_1$ and $w_2 = x_2 + y_2$ for $x_1, x_2 \in N$ and $y_1, y_2 \in K$. Now,
$w_1+ w_2 = (x_1 + y_1) + (x_2 + y_2) = (x_1+ x_2) + (y_1+ y_2) \in N+K$.

ii. let $w= x + y \in N+K$, $r \in R$. so, $rw = r(x+y) = rx + ry \in N+K$.

5. let N_α; $\alpha \in I$(I is the index set), be a family of submodules of an R-module M, then $\bigcap_{\alpha \in I} N_\alpha$ is also a submodule of M.
   Proof. H.W.

6. let N be a submodule of an R-module M and $\frac{M}{N}$ = {m+N| m∈M}.
   clearly that ($\frac{M}{N}$,+) is an abelian group where for each m, m_1, m_2∈ M, r ∈ R:
   i. $(m_1+N) + (m_2+N) = (m_1+ m_2) +N$
   ii. and $r.(m_2+N) = (r.\ m_2)+ M$.
   then $\frac{M}{N}$ is an R-module, which is called the ***quotient module*** of M by N.

Remark. ***Modular Law.*** There is one property of modules that is often useful. It is known as the modular law or as the modularity property of modules. If N , L and K are modules, then N∩(L+K) = (N∩L)+(N∩K). If N , L and K are submodules of an R-module M and L ≤ N, then N∩(L+K) = L + (N∩K).

Definition. Let M be an R-module. If there exists $x_1, x_2, \ldots, x_n \in M$ such that $M= Rx_1+ Rx_2+ \ldots +Rx_n$. M is said to be ***finitely generated*** module. If M = Rx=<x> = {rx | r ∈ R} is said to be ***cyclic*** module.
Examples.
1. $\mathbb{Z}_n = <\bar{1}>$ is cyclic $\mathbb{Z}$-module for all n ∈ $\mathbb{Z}$.
2. $n\mathbb{Z} = < n >$ is cyclic $\mathbb{Z}$-module for all n ∈ $\mathbb{Z}$.
3. If F is any field, then the ring F[x,y] has the submodule(ideal) <x,y> which is not cyclic.
4. Q is not finitely generated $\mathbb{Z}$-module.

Direct sums and products

Definition. Let R be a ring and {M_i| i ∈I } be an arbitrary (possibly infinite) of a nonempty family of R-modules. $\prod_{i\in I} M_i$ is the ***direct product***

of the abelian groups M_i, and $\bigoplus_{i\in I} M_i$ the ***direct sum*** of the of the abelian groups M_i, where

$$\prod_{i\in I} M_i = \{f: I \to \bigcup_{i\in I} M_i \mid f(i) \in M_i, \text{ for all } i\in I\}$$

Define a binary operation "+" on the direct product (of modules) $\prod_{i\in I} M_i$ as follows: for each f,g $\in \prod_{i\in I} M_i$ (that is, f,g : $I \to \bigcup_{i\in I} M_i$ and f(i),g(i) $\in M_i$ for each i), then f+g : $I \to \bigcup_{i\in I} M_i$ is the function given by $i \to f(i)+g(i)$.

i.e $\quad$ $(f+g)(i) = f(i)+g(i)$ $\quad$ for each $i \in I$.

Since each M_i is a module, $f(i)+g(i) \in M_i$ for every i, whence $f+g \in \prod_{i\in I} M_i$.
So ($\prod_{i\in I} M_i$, +) is an abelian group

Now, if $r \in R$ and $f \in \prod_{i\in I} M_i$, then rf : $I \to \bigcup_{i\in I} M_i$ as (rf)(i) = r(f(i)).

1. $\prod_{i\in I} M_i$ is an ***R-module*** with the action of R given by r(f(i)) = (rf(i)) (i.e define α: R x $\prod_{i\in I} M_i \to \prod_{i\in I} M_i$ by α(r,f) = rf)
2. $\bigoplus_{i\in I} M_i$ is a ***submodule*** of $\prod_{i\in I} M_i$. (H.W.)

Remark. $\prod_{i\in I} M_i$ is called the (external) direct product of the family of R-modules $\{M_i \mid i \in I\}$ and $\bigoplus_{i\in I} M_i$ is (external) direct sum. If the index set is finite, say i = { 1 ,2, … , n}, then the direct product and direct sum coincide and will be written $M_1 \oplus M_2 \oplus \ldots \oplus M_n$.

Definition. ((internal) direct sum) Let R be a ring and N, K submodules of an R-module M such that:

1. M = N + K
2. N ∩ K = 0

Then N and K is said to be ***direct summand*** of M and M = N⊕ K ***internal direct sum*** of N and K.

Definition. Let R be an integral domain. An element x of an R-module M (x∈ M) is said to be ***torsion*** element of M if $\exists(0\neq)$ r ∈ R with rx = 0.

Example.

1. Let M = $\mathbb{Z}_6$ as $\mathbb{Z}$-module. Then every element in $\mathbb{Z}_6$ is torsion: $\bar{3} \in \mathbb{Z}_6$, $\exists\ 2 \in \mathbb{Z}$ such that $2.\ \bar{3} = \bar{0}$

$\bar{2} \in \mathbb{Z}_6$, $\exists\, 3 \in \mathbb{Z}$ such that $3.\,\bar{2} = \bar{0}$
$\bar{1} \in \mathbb{Z}_6$, $\exists\, 6 \in \mathbb{Z}$ such that $6.\,\bar{1} = \bar{0}$
$\bar{4} \in \mathbb{Z}_6$, $\exists\, 3 \in \mathbb{Z}$ such that $3.\,\bar{4} = \bar{0}$
$\bar{5} \in \mathbb{Z}_6$, $\exists\, 6 \in \mathbb{Z}$ such that $6.\,\bar{5} = \bar{0}$

2. Every element in $\mathbb{Z}_n$ as $\mathbb{Z}$-module is torsion.
3. The only torsion element in M = Q as $\mathbb{Z}$-module is zero (if $(0\neq)$ $x \in Q$, then $\nexists$ $(0\neq)$ $r \in \mathbb{Z}$ such that $rx = 0$.

Remark. Let M be an R-module where R is an integral domain, then the set of all torsion elements of M, denoted by $\tau(M)$ is a submodule of M ($\tau(M) = \{x \in M \mid \exists (0\neq)\, r \in R \text{ such that } rx = 0\}$)

Proof. 1. $\tau(M) \neq \varphi$ $(0 \in \tau(M))$

2. if $x, y \in \tau(M)$, then $\exists(0\neq)$ $r_1, r_2 \in R$ such that $r_1x = 0$ and $r_2y = 0$. Since R is an integral domain, $r_1 \neq 0$ and $r_2 \neq 0$, so $r_1.\, r_2 \neq 0$. Hence $r_1.r_2(x+y) = r_1.r_2\, x + r_1.r_2y = r_2.r_1\, x + r_1.r_2y = 0 + 0 = 0$. Thus $x+y \in \tau(M)$

3. let $(0\neq)$ $r \in R$ $w \in \tau(M)$, $\exists(0\neq)$ $r_1 \in R$ with $r_1w = 0$. Now, $r_1(rw) = 0$ implies $rw \in \tau(M)$.

$\therefore$ $\tau(M)$ is a submodule of M.

Remark. In general, If R is not integral domain, then $\tau(M)$ may not submodule of M in general.

Definition. Let M be a module over integral domain R. If $\tau(M) = 0$, Then M is said to be ***torsion free*** module. If $\tau(M) = M$, then M is said to be ***torsion*** module.

Examples. 1. The $\mathbb{Z}$-module Q, is torsion free module.
2. The $\mathbb{Z}$-module $\mathbb{Z}_n$, is torsion module.

Remark. Let M be a module over an integral domain R, then $\frac{M}{\tau(M)}$ is torsion free R-module. (i.e $\tau(\frac{M}{\tau(M)}) = \tau(M)$)

Proof. Let m+ τ(M) $\in \tau(\frac{M}{\tau(M)})$, $\exists(0\neq)$ r $\in$ R such that r(m + τ(M)) = τ(M).
$\rightarrow$ rm + τ(M) = τ(M) $\rightarrow$ rm $\in \tau$(M)
$\rightarrow \exists(0\neq)$ s $\in$ R such that s(rm) = (sr)m = 0
$\because$ sr $\neq$ 0 $\rightarrow$ m $\in \tau$(M) $\rightarrow$ m + τ(M) = τ(M) $\rightarrow \tau(\frac{M}{\tau(M)}) = \tau$(M).

Exercises.

1. Every submodule of torsion module over integral domain is torsion module.
2. Every submodule of torsion free module over integral domain is torsion free module.

Definition. Let M be a module over an integral domain R. An element x∈M is said to be ***divisible*** element if for each $(0\neq)$ r $\in$ R $\exists$y $\in$ M such that ry = x.

Examples.

1. 0 is divisible element in every module M.
2. Every element in a $\mathbb{Z}$-module Q is divisible element.
3. 0 is the only divisible element in $2\mathbb{Z}$ as $\mathbb{Z}$-module.

Remark. Let M be a module over an integral domain R. the set of all divisible element of M denoted by ∂(M) = {m $\in$ M| $\forall$ $(0\neq)$ r $\in$ R, $\exists$ y $\in$ M such that m = ry}

Definition. Let M be a module over an integral domain R. M is said to be ***divisible*** module if ∂(M) = M.

Examples.

1. The $\mathbb{Z}$-module $\mathbb{Z}$ is not divisible.
2. The module Q over the ring $\mathbb{Z}$ is divisible.
3. The $\mathbb{Z}$-module $\mathbb{Z}_n$ is not divisible.

Proposition. Let R be an integral domain and M R-module. Then:

1. ∂(M) is a submodule of M.

2. If M is divisible module, then so is $\frac{M}{\mathrm{N}}$ for all submodule N of M.
3. M is divisible module iff M = rM for all $0 \neq r \in R$.
4. If $M = M_1 \oplus M_2$, then $\partial(M) = \partial(M) \oplus \partial(M)$.

Proof. 1. Let $x,y \in \partial(M)$, then

$\forall\ 0 \neq r \in R$, $\exists\ x_1 \in M$ such that $x = rx_1$
$\forall\ 0 \neq r \in R$, $\exists\ y_1 \in M$ such that $y = ry_1$
i) $x + y = r(x_1+y_1)$, for all $0 \neq r \in R$. implies $x+y \in \partial(M)$.

ii) let $x \in \partial(M)$ and $0 \neq s \in R$, then $\forall\ 0 \neq r \in R$, $\exists\ y \in M$ such that $x = ry$.
Since R is an integral domain, $r \neq 0$ and $s \neq 0$, then $rs \neq 0$.
So $sx = s(ry) = (sr)y$. implies that $sx \in \partial(M)$.

$\therefore \partial(M)$ is a submodule of M

2. Let $x + N \in \frac{M}{N}$ where $x \in M$. Since M is divisible and $x \in M$, then for $\forall\ 0 \neq r \in R$, $\exists\ y \in M$ such that $x + N = ry + N = r(y+N)$.

$\therefore \frac{M}{N}$ is divisible module

3. $\rightarrow$)Suppose that M is divisible module. To prove M = Rm, must prove that: a. $M \leq rM$ b. $rM \leq M$
for that :
a. Let $m \in M$. Since $M = \partial(M)$ (M is divisible), so $m \in \partial(M)$.
For all $0 \neq r \in R$, $\exists\ n \in M$ such that $m = rn \in rM$. Hence $M \leq rM$.

b. Since M is a module then $rM \leq M$.

$\therefore M = rM$

$\leftarrow$) Suppose that $M = rM$ for all $0 \neq r \in R$. if $m \in M = rM$, then $m = rn$ for $n \in M$ and all $0 \neq r \in R$. implies that $m \in \partial(M)$. Thus $M \leq \partial(M)$.
let $x \in \partial(M)$, $\forall\ 0 \neq r \in R$, $\exists\ y \in M$ such that $x = ry$. Thus $\partial(M) \leq M$. Hence $M = \partial(M)$. So M is divisible module.

Remark. Point (2) in the previous proposition means: the quotient of divisible module is divisible.

<u>Exercise</u>. Is every submodule of divisible module divisible?

Definition. Let M be an R-module and $x \in M$. Then the set

$$\mathbf{ann_R(x)} = \{r \in R \mid rx = 0\}$$

is said to be **annihilator of the element x in R**.

Remarks.

1. Let M be an R-module and $x \in M$. Then the set
$$\mathbf{ann_R(M)} = \{r \in R \mid rM = 0\}$$
$$= \{ r \in R \mid rm = 0 \text{ for all } m \in M\}$$
   is said to be **annihilator of the module M in R**.
2. Let M be an R-module. If $ann_R(M) = 0$, then M is said to be **faithful** module.

Examples.

1. The $\mathbb{Z}$-module $\mathbb{Z}$ is faithful ($ann_{\mathbb{Z}}(\mathbb{Z}) = 0$)
2. The $\mathbb{Z}$-module Q is faithful ($ann_{\mathbb{Z}}(Q) = 0$)
3. The $\mathbb{Z}$-module $\mathbb{Z}_n$ is not faithful ($ann_{\mathbb{Z}}(\mathbb{Z}_6) = 6\mathbb{Z}$)
4. $ann_{\mathbb{Z}_6}(\{\bar{0}, \bar{3}\}) = \{\bar{0}, \bar{2}, \bar{4}\}$
5. $ann_{\mathbb{Z}}(\{\bar{0}, \bar{3}\}) = 2\mathbb{Z}$
6. $ann_{\mathbb{Z}}(\{\bar{0}, \bar{2}, \bar{4}\}) = 3\mathbb{Z}$
7. $ann_{\mathbb{Z}_6}(\{\bar{0}, \bar{2}, \bar{4}\}) = \{\bar{0}, \bar{3}\}$
8. $ann_{\mathbb{Z}_n}(\mathbb{Z}_n) = n\mathbb{Z}$

Definition. Let N and K be submodules of an R-module M. The set

$$(N: K) = \{r \in R \mid rK \leq N\}$$

is an ideal of R which is called residual.

Remark.

1. If N = 0, then

$$(0: K) = \{r \in R \mid rK = 0\} = ann_R(K)$$

2. If N = 0 and K = M, then

$$(0: M) = \{r \in R \mid rM = 0\} = ann_R(M)$$

Chapter two (Module homomorphisms)

Definition. Let M and N be modules over a ring R . A function $f : M \to N$ is an ***R-module homomorphism*** (simply homomorphism) provided that for all $x, y \in M$ and $r \in R$:

1. $f(x+y) = f(x) + f(y)$
2. $f(rx) = rf(x)$.

If R is a field ring, then an R-module homomorphism is called a ***linear transformation***.

Remarks.

1. if f is injective and homomorphism, then is said to be monomorphism.
2. if f is surjective and homomorphism, then is said to be epimorphism.
3. if f is injective, surjective and homomorphism, then is said to be isomorphism (and written $M \approx N$) .

Examples.

1. $2\mathbb{Z}_{\mathbb{Z}} \approx 3\mathbb{Z}_{\mathbb{Z}}$.

Proof. Define $g: 2\mathbb{Z} \to 3\mathbb{Z}$ as $g(2n) = 3n$ for all $n \in \mathbb{Z}$.

i. g is well-define.

ii. g is homomorphism : for $2n, 2n_1, 2n_2 \in 2\mathbb{Z}$, $r \in \mathbb{Z}$

$g(2n_1 + 2n_2) = g(2(n_1 + n_2)) = 3(n_1+n_2) = 3n_1 + 3n_2 = g(2n_1)+g(2n_2)$

$g(r(2n)) = g(2rn) = 3rn = r(3n) = rg(2n)$

iii. g is one – to – one. If $g(2n_1) = g(2n_2)$, then

$\rightarrow\ 3n_1 = 3n_2 \ \rightarrow\ n_1 = n_2 \ \rightarrow\ 2n_1 = 2n_2$.

iv. g is onto. for all $y = 3n \in 3\mathbb{Z}$, there is $x = 2n \in 2\mathbb{Z}$ such that $g(2n) = 3n$.

Hence $2\mathbb{Z} \approx 3\mathbb{Z}$(i.e g is an isomorphism).

2. Let R be a ring and $\{ M_i \mid i \in I \}$ a family of submodules of an R-module M such that:
   i. M is the sum of the family $\{ M_i \mid i \in I \}$
   ii. for each $k \in I$, $M_k \cap \sum_{i\in I, i\neq k} M_i = 0$

   Then $M \approx \bigoplus_{i\in I} M_i$

   (Hint : define $\beta: \bigoplus_{i\in I} M_i \rightarrow M$ by $\beta(f) = \sum_{i\in I} f(i)$)
3. Let $\{ M_i \mid i \in I \}$be family of R-modules.
   i. For each $k \in I$, the canonical projection $\rho_k: \prod_{i\in I} M_i \rightarrow M_k$ defined by $\rho_k(f) = f(k)$ is an R- module epimorphism .
   ii. For each $k \in I$, the canonical injection $J_k: M_k \rightarrow \prod_{i\in I} M_i$ defined by for $x \in M_k$, $(J_k(x))i = \begin{cases} x & if\ i = k \\ 0 & otherwise (i \neq k) \end{cases}$

   is an R-module monomorphism.

   iii. $\rho_k o J_k = I_{M_k}$.

   Proof. $\rho_k o J_k : M_k \rightarrow M_k$ with $(\rho_k o J_k)(x) = \rho_k(J_k(x)) = J_k(x)(k) = x$

   iv. $J_k\, o\, \rho_k \neq I_{M_k}$.
4. Let K be a submodule of a module M. the function $\pi: M \rightarrow \frac{M}{K}$ defined by $\pi(x) = x+K$ for all $x \in M$, is an R-homomorphism and onto. This homomorphism is called the natural epimorphism.

<u>**Exercises.**</u> Prove :

1. If R is a ring, the map $R[x] \rightarrow R[x]$ given by $f \rightarrow f(x)$(for example, $(x^2 + 1) \rightarrow x(x^2 + 1)$) is an R-module homomorphism, **but not** a ring homomorphism (prove that).
2. $Hom(R, M) \approx M$
3. for each $n \in \mathbb{Z}$, $\frac{\mathbb{Z}}{n\mathbb{Z}} \approx \mathbb{Z}_n$.

Theorem. Let f : M → N be a homomorpism, then

1. ***kernel of f*** (kerf = {x ∈M| f(x) = 0}) is a submodule of M.
2. ***Image of f*** (Imf={n ∈N| n = f(m) for some m ∈ M}) is a submodule of N.
3. f is a monomorpism iff kerf = 0.
4. f : M→N is an R-module isomorphism if and only if there is A homomorphism g : N →M such that gf = I_M and fg = I_N.

Proof. H.W.

Proposition. Let R be an integral domain and M be an R-module, then:

1. If f : $M \rightarrow \acute{M}$ be a module homomorphism, then f(τ(M)) ≤ $\tau(\acute{M})$.
2. If M = $M_1 \oplus M_2$, then τ(M) = $\tau(M_1) \oplus \tau(M_2)$.

Definition. An R-module, M is called ***simple*** if M ≠ {0} and the only submodules of M are M and {0}

Proposition. Every simple module M is cyclic (i.e M = Rm for every nonzero m ∈ M).

Proof. Let M be a simple R-module and m ∈ M. Both Rm and B = { c ∈ M| Rc = 0 } are submodules of M. Since M is simple, then each of them is either 0 or M. But RM ≠ 0 implies B ≠ M. Consequently B = 0, whence Ra = M for all nonzero m ∈ M. Therefore M is cyclic

Remark. The converse is not true in general: that is a cyclic module need not be simple for example, the cyclic Z-module Z_6.

Examples.

1. The $\mathbb{Z}$-module $\mathbb{Z}_3$ is simple.
2. The $\mathbb{Z}$-module $\mathbb{Z}_p$ is simple for each prime integer's p.
3. The $\mathbb{Z}$-module $\mathbb{Z}_4$ is not simple, since the submodule$\{\bar{0}, \bar{2}\} \neq 0$ and $\{\bar{0}, \bar{2}\} \neq \mathbb{Z}_4$.
4. The $\mathbb{Z}$-module $\mathbb{Z}$ is not simple.(why?)
5. Every division ring D is a simple ring and a simple D-module

Lemma. (**Schur's lemma**)

1. Every R-homomorphism from a simple R-module is either zero or monomorphism.
2. Every R-homomorphism into a simple R-module is either zero or epimorphism.
3. Every R-homomorphism from a simple R-module into simple R-module is either zero or isomorphism.

Proof 1. Let M be a simple module and f: M→ N be an R-module homomorphism. Then kerf is a submodule of M . But M is simple.
So either kerf = {0} , implies f is one-to-one
or kerf = M, implies f is zero homomorphism.

Proof 2. Let N be a simple module and f: M→ N be an R-module homomorphism. Then Imf is a submodule of N . But N is simple.
So either Imf = {0} , implies f zero homomorphism
or Imf = N, implies f is onto.

Proof 3. as a consequence to (1) and (2), the proof of (3) holds.

Examples. 1. An R-module homomorphism f: $\mathbb{Z}_4 \rightarrow \mathbb{Z}_5$ is zero.

2. An R-module homomorphism f: $\mathbb{Z}_3 \rightarrow \mathbb{Z}_5$ is zero.

Exercise. Let M ≠ {0} be an R-module. Prove that:
If N_1, N_2 are submodules of M, with N_1 simple and $N_1 \cap N_2 \neq 0$, then $N_1 \leq N_2$

Remark. Let A, B be two simple R-module, then Hom(A, B) is either zero or for all f ∈ Hom(A, B) is an isomorphism, where Hom(A, B) = {f:A→B| f is homomorphism}

Isomorphism theorems

First isomorphism theorem. Suppose f: M→ N is an R-module homomorphism. Then $\frac{M}{kerf} \approx$ f(M) .

Proof. Define h : $\frac{M}{kerf}$ → f(M) by: h(m + kerf) = f(m) for all m ∈ M.

1. h is well define: Let m_1+ kerf , m_2 + kerf ∈ $\frac{M}{kerf}$ such that

$$m_1 + kerf = m_2 + kerf \text{ implies } m_1 - m_2 \in kerf$$

and so

$$f(m_1 - m_2) = f(m_1) - f(m_2) = 0 \rightarrow f(m_1) = f(m_2)$$

Hence

$$h(m_1 + kerf) = h(m_2 + kerf)$$

$$\therefore \text{ h is well define}$$

2. h is a homomorphism since f is homomorphism.
3. h is a monomorphism: for that suppose that

$$h(m_1 + kerf) = h(m_2 + kerf).$$

from definition of h, $f(m_1) = f(m_2)$ implies $f(m_1) - f(m_2) = f(m_1 - m_2)=0$
so $m_1 - m_2 \in kerf \rightarrow m_1 + kerf = m_2 + kerf$

4. h is an epimporphism: let y∈ f(m) ∈ f(M), ∃ m + kerf ∈ $\frac{M}{kerf}$ such that

   h(m+ kerf) = f(m) = y

$$\therefore \text{ h is an epimorphism}$$

So h is an isomorphism and by this, $\frac{M}{kerf} \approx f(M)$

Remark. If f is an epimorphism, then $\frac{M}{kerf} \approx N$

<u>Second isomorphism theorem.</u> Let N and K be submodules of an R-module M, then $\frac{K+N}{N} \approx \frac{K}{N \cap K}$

Proof. Define α: $K \rightarrow \frac{K+N}{N}$ by α(x) = x + N for each x ∈ K.

1. α is well-define (prove)
2. α is homomorphism (prove)
3. α is epimorphism (prove)
4. kerα = { x ∈ K| α(x) = 0}
   ={ x ∈ K | x + N = N}
   ={ x ∈ K | x ∈ N}
   = N∩K

Then by the first isomorphism theorem, $\frac{K}{N \cap K} \approx \frac{K+N}{N}$

<u>Third isomorphism theorem</u>. Let N, K be submodules of M, and K≤ N,

then $\frac{\frac{M}{K}}{\frac{N}{K}} \approx \frac{M}{N}$.

Proof. Define g: $\frac{M}{K} \to \frac{M}{N}$ by :g(m + K) = m + N for all m ∈ M.

1. g is well-define:
   suppose m_1+ k = m_2+K iff $m_1 - m_2 \in K \leq N$ iff $m_1 + N = m_2 + N$
   ∴ g is well defined
2. g is a homomorphism (prove)
3. g is an epimorphism (prove)
4. kerg = {m+K| g(m+ k) = N}
   ={m+K| m+ N = N}
   = {m+K| m∈ N}
   = $\frac{N}{K}$ (where K ≤ N and m ∈ N)
   ∴ kerg = $\frac{N}{K}$

Then by the first isomorphism theorem, $\frac{\frac{M}{K}}{\frac{N}{K}} \approx \frac{M}{N}$.

<u>Exercise</u>. Let M be a cyclic R-module, say M=Rx. Prove that M≈R/ ann(x), where ann(x) = {r ∈ R| rx = 0}.

[Hint: Define the mapping f: R→M by f(r) = rx]

Chapter three (Sequence)

Short exact sequence

Definition. A sequence $M_1 \xrightarrow{f} M \xrightarrow{g} M_2$ of R-modules and R-module homomorphismsis said to be ***exact** at* M Im f = ker g while a sequence of the form

$$\partial: \quad \ldots \to M_{n-1} \xrightarrow{f_{n-1}} M_n \xrightarrow{f_{n+1}} M_{n+1} \to \cdots$$

n$\in \mathbb{Z}$, is said to be an *exact sequence* if it is exact at M_n for each n $\in \mathbb{Z}$. A sequence such as

$$0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$$

that is exact at M_1, at M and at M_2 is called a ***short exact sequence***.

Remarks.

1. If an exact sequence $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ is short exact then
   i. f is a monomorphism
   ii. g is an epimorphism
2. A sequence $0 \to M_1 \xrightarrow{f} M$is exact iff f is monomorphism
3. A sequence $M \xrightarrow{g} M_2 \to 0$ is exact iff g is epimorphism
4. If the composition(between two homomorphisms f and g) gof = 0, then Imf $\leq$ kerg.

Examples.

1. If N is a submodule of M, then $0 \to N \xrightarrow{i} M \xrightarrow{\pi} \frac{M}{N} \to 0$ is a short exact sequence, where i is the canonical injection and π is the natural epimorphism. for example : since kerf is a submodule of M, then $0 \to kerf \xrightarrow{i} M \xrightarrow{\pi} \frac{M}{kerf} \to 0$ is a short exact sequence.
2. Consider the sequence

$$\mu: \quad 0 \to M_1 \xrightarrow{J_1} M_1 \oplus M_2 \xrightarrow{\rho_2} M_2 \to 0$$

$\mathrm{Im}J_1 = M_1 \oplus \{0\}$; $J_1(x) = (x, 0)$
$\ker\rho_2 = M_1 \oplus \{0\}$; $\rho_2(x, y) = (0, y)$
for any $x \in M_1$, $y \in M_2$ and $(x,y) \in M_1 \oplus M_2$
J_1 is a monomorphism and ρ_2 is an epimorphism
$\therefore \mu$ is short exact sequence

3. The sequence $0 \to 2\mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\pi} \frac{\mathbb{Z}}{2\mathbb{Z}} \to 0$ of $\mathbb{Z}$-modules is a short exact sequence

Remark. ***Commutative Diagrams***
The following diagram

$$\begin{array}{ccc} A & \xrightarrow{f_1} & B \\ g_1\downarrow & & \downarrow g_2 \\ C & \xrightarrow{f_2} & D \end{array}$$

is said to be ***commutative*** if $g_2 o f_1 = f_2 o g_1$.Similarly, for a diagram of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

is commutative if gof = h and we say that g ***completes the diagram commutatively***.

Theorem. **(*The short five lemma*).** Let R be a ring and

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{f_1} & B & \xrightarrow{g_1} & C & \to & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\ 0 & \to & \acute{A} & \xrightarrow{f_2} & \acute{B} & \xrightarrow{g_2} & \acute{C} & \to & 0 \end{array}$$

a commutative diagram of R-modules and R-module homomorphisms such that each row is a short exact sequence. Then

1. If α and γ are monomorphisms, then β is a monomorphism.
2. If α and γ are epimorphisms, then β is an epimorphism.
3. if α and γ are isomorphisms, then β is an isomorphism.

Proof 1.

To show that β is a monomorphism, must prove ker β = 0.

Let b ∈ ker β → β(b) = 0 →g_2(β(b)) = g_2(0) = 0. Since the diagram is commutative, then:

γog_1(b) = γ(g_1(b)) = 0 → g_1(b) ∈ kerγ = {0}(γ is a monomorphism) →g_1(b) = 0 → b ∈ kerg_1= Imf_1= f_1(A). There is a ∈ A such that

$$f_1(a) = b \rightarrow \beta(f_1(a)) = \beta(b).$$

Since

βof_1 = f_2oα → f_2oα(a) = β(b) → f_2(α(a)) = 0 → α(a) ∈ kerf_1= {0}(f_2 is a monomorphism), so

$$\alpha(a) = 0 \rightarrow a \in \ker\alpha = \{0\} \ (\alpha \text{ is a monomorphism}) \rightarrow a = 0.$$

But f_1(a)=b and a=0 → b=f_1(a)= f_1(0)= 0 → b=0.

$$\ker\beta = \{0\} \rightarrow \beta \text{ is a monomorphism}$$

Proof 2.

Let $\acute{b}$ ∈ $\acute{B}$ → $g_2(\acute{b})$ ∈ $\acute{C}$ → $g_2(\acute{b})$ = $\acute{c}$. Since γ is an epimorphism, there is c ∈ C such that

$$\gamma(c) = \acute{c} \rightarrow g_2(\acute{b}) = \gamma(c).$$

But g_1 is an epimorphism, then there is b ∈ B such that

$$g_1(b) = c \rightarrow g_2(\acute{b}) = \gamma(c) = \gamma(g_1(b)) = \gamma o g_1(b) = g_2 o\beta(b)$$

so

$$g_2(\acute{b}) = g_2(\beta(b)) \rightarrow g_2(\beta(b) - \acute{b}) = 0 \ (g_2 \text{ is homomorphism}).$$

and

$$\beta(b) - \acute{b} \in \ker g_2 = \mathrm{Im} f_2 \rightarrow \beta(b) - \acute{b} \in \mathrm{Im} f_2.$$

There is $\acute{a}$ ∈ $\acute{A}$ such that $f_2(\acute{a})$ = β(b) - $\acute{b}$. But α is an epimorphism, there is a ∈ A such that α(a) = $\acute{a}$. Since βof_1 = f_2oα (the diagram is commutative).

Then

$$\beta\,(f_1(a)) = f_2(\alpha(a)) = f_2(\acute{a}) = \beta(b) - \acute{b}$$

so

$$\acute{b} = \beta(b) - \beta\,(f_1(a)) = \beta(b - f_1(a)) \text{ (}\beta\text{ is homomorphism)}$$

i.e there is $b - f_1(a) \in B$ such that $\beta(b - f_1(a)) = \acute{b}$

Hence β is an epimorphism.

Proof 3. is an immediate consequence of (1) and (2).

Exercise. Consider the following diagram:

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "h"'] & C \arrow[r] & 0 \\ & D & & \end{tikzcd}$$

where the row is exact and hof = 0. Prove that, there exact a unique homomorphism k: C ⟶ D such that kog = h.

Definition. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence. This sequence is said to be ***splits*** if Imf is a direct summand of B.
(i.e there is $D \leq B$ such that $B = \mathrm{Im}f \oplus D$).

Example. The sequence $0 \to 2\mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\pi} \frac{\mathbb{Z}}{2\mathbb{Z}} \to 0$ of ℤ-modules and ℤ-homomorphism is a short exact sequence which is not split (where Imi = 2 ℤ is not direct summand of ℤ).

Theorem. Let R be a ring and

$$\mathcal{F}: \quad 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

a short exact sequence of R-module homomorphisms. Then the following conditions are equivalent

1. $\mathcal{F}$ splits.
2. f has a left inverse (i.e ∃ h: B ⟶ A homomorphism with hof = I_A).
3. g has a right inverse(i.e ∃ k:C⟶B a homomorphism with gok = I_C).

Proof. (1 ⟶ 2) since $\mathcal{F}$ splits, then Imf is a direct summand of B.
(i.e. ∃ $B_1 \leq B$ such that $B = \mathrm{Im}f \oplus B_1$).

Define h: $B \to A$ by $h(x) = h(a_1+b_1) = a$ for $x = a_1+b_1 \in Imf \oplus B_1$.
where $a_1 \in Imf$ (i.e $\exists$ $a \in A$ such that $f(a) = a_1$) and $b_1 \in B_1$.

a. Since f is one-to-one, then h is well-define.
b. h is a homomorphism
c. let $w \in A$, $hof(w) = h(f(w)) = h(f(w)+ 0) = w$ (by definition of h)

$\therefore$ h is a left inverse of f.

$(2 \to 3)$ suppose f has a left inverse say h(i.e. $hof = I_A$).
Define k: $C \to B$ by: $k(y) = b - foh(b)$ where $g(b) = y$ with $b \in B_1$.

a. k is well define:
   let $y, y_1 \in C$ such that $y = y_1$ with $g(b) = y$ and $g(b_1) = y_1$ for $b, b_1 \in B_1$.
   Now,

$$g(b) = g(b_1) \to b_1 - b \in \ker g = Imf$$

   so, $b_1\text{-}b \in Imf \to \exists\ a \in A$ such that $f(a) = b_1 - b$.
   Then $h(f(a)) = h(b_1- b)$. But $hof = I_A$,
   so $a = hof(a) = h(f(a)) = h(b_1- b) = h(b_1) - h(b)$
   $\therefore a = h(b_1) - h(b) \to f(a) = f(h(b_1)) - f(h(b)) = b_1 - b$
   $\therefore$ $b - f(h(b)) = b_1- f(h(b_1)) \to k(y) = k(y_1) \to$ k is well define.
b. k is homomorphism (why?)
c. $gok = I_C$. for that
   let $y \in C$, $gok (y) = g(k(y)) = g(b\text{-}foh(b))$ where $g(b) = y$.
   $\to gok(y) = g(b) + gofoh(b)$. But Im f = kerg. So, $gof = 0$.
   $\to gok(y) = g(b) + 0 = y$

$\therefore gok = I_C$

$(3 \to 1)$ suppose that g has a right inverse say k: $C \to B$ such that $gok = I_C$
Let $B_1 = \{b \in B|\ kog(b) = b\}$

a. $B_1 \neq \varphi$ ($0 \in B_1$ where $g(0) = k(g(0)) = k(0) = 0$)
b. B_1 is a submodule of B. (prove?)
c. $B = Imf \oplus B_1$, for that:
   i. Let $w = Imf \cap B_1 \to w = f(a) \in B_1$ for some $a \in A$ with $kog(w)=w \to k(g(f(a))) = k(0) =0$. But $k(g(f(a))) = k(g(w)) = w$.
   Thus $w = 0$ and so $Imf \cap B_1 = 0$.
   ii. Let $b \in B \to b = b - kog(b) + kog(b)$.

Since $kog(kog(b)) = kog(b)$, then $kog(b) \in B_1$ and $g(b\text{-}kog(b)) = g(b) - gokog(b) = g(b) - Iog(b) = g(b) - g(b) = 0$ (where $gok = I_C$).
$\rightarrow b\text{-}kog(b) \in kerg = Imf$
$\therefore b = b\text{-}kog(b) + kog(b) \in Imf + B_1$
$\therefore B = Imf \oplus B_1 \rightarrow$ Imf is a direct summand of B which implies $\mathcal{F}$ splits.

Exercise
If the short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

splits, then $B \approx Imf \oplus Img$

Chapter four (Noetherian and Artinian modules)

Ascending and Descending chain condition

Definition. An R-module M is said to be satisfy the ascending chain condition (resp. descending chain condition) if for every ascending (resp. descending) chain of submodules

$$M_1 \leq M_2 \leq M_3 \leq \ldots \leq M_n \leq \ldots$$

$$(\text{resp.} \quad M_1 \geq M_2 \geq M_3 \geq \ldots \geq M_n \geq \ldots)$$

there exists $m \in \mathbb{Z}_+$ such that $M_n = M_m$ whenever $n \geq m$.

Definition. A module which satisfies the ascending chain condition is said to be ***Noetherian***.

Definition. A module which satisfies the descending chain condition is said to be ***Artinian***.
Remark. A ring R is said to be *Noetherian* (*Artinian)* if it is *Noetherian* (*Artinian*) as an R-module. i.e., if it satisfies a.c.c. (d.c.c.) on ideals.

Example. Every simple module is both Noetherian and Artinian.

Theorem 1. Let M be an R-module. Then the following statements are equivalent:

1. M satisfies the ascending (descending) chain condition.
2. For any nonempty family $\{M_\alpha\}_{\alpha\in I}$ of submodules of M, there exist a maximal (minimal) element M_0 satisfies the maximal condition (resp. minimal condition)
   (i.e $\exists M_0 \in \{M_\alpha\}_{\alpha\in I}$ such that whenever $M_0 \leq M_\beta$, then $M_0 = M_\beta$)
   (resp. i.e $\exists M_0 \in \{M_\alpha\}_{\alpha\in I}$ such that whenever $M_\beta \leq M_0$, then $M_0 = M_\beta$)

Proof. (1→2) consider the set

$$\mathcal{F} = \{M_i \mid M_i \leq M\}$$

$\mathcal{F} \neq \varphi$

Suppose $\mathcal{F}$ has no maximal element.

Let $M_1 \in \mathcal{F}$ implies M_1 is not maximal element.

$\exists\ M_2 \in \mathcal{F}$ such that $M_1 \leq M_2$. Since M_2 is not max. element, then there is $M_3 \in \mathcal{F}$ such that $M_2 \leq M_3$.

Continuing in this way, we get

$$M_1 \leq M_2 \leq M_3 \leq \dots$$

A chain of submodules of M. if this sequence is an infinite, then it does not satisfy the ACC. C!

$$\therefore\ \mathcal{F} \text{ has maximal element}$$

(2 → 1) suppose M satisfies the maximal condition for submodules, and let

$$M_1 \leq M_2 \leq M_3 \leq \dots$$

be ascending chain of submodules of M.

Let $\mathcal{H} = \{M_\alpha\}_{\alpha\in I}$ be a family of the submodules of M. Then $\mathcal{H} \neq \varphi$ and has maximal element M_m. implies whenever $n \geq m$, $M_m = M_n$.

$$\therefore \mathcal{H} \text{ satisfies the ascending chain condition.}$$

Theorem 2. Let M be an R-module. Then the following statements are equivalent:

1. M is Noetherian.
2. Every submodule of M is finitely generated.

Proof.

(1 → 2) suppose M is Noetherian module and K be submodule of M.

Let $\mathcal{F}$ = {A| A is finitely generated submodule of K}

$\mathcal{F} \neq \varphi$ (the zero submodule of A is in $\mathcal{F}$)

Since M is Noetherian module , so $\mathcal{F}$ has maximal element say K_0.

Hence K_0 is finitely generated submodule of K

$$\text{i.e } K_0 = Rk_1 + Rk_2 + \ldots + Rk_n$$

Suppose $K_0 \neq K \rightarrow \exists\ a \in K$ and $a \notin K_0$ and so

$$K_0 + Ra = K_0 = Rk_1 + Rk_2 + \ldots + Rk_n + Ra$$

∵ $K_0 + Ra$ is a finitely generated submodule of K, then $K_0 + Ra \in \mathcal{F}$ is a contradiction with the maximalist of K_0. Hence $K_0 = K$

∴ K is a finitely generated

(2 → 1) suppose that every submodule of M is finitely generated.

Let $K_1 \leq K_2 \leq K_3 \leq \ldots$ be an ascending chain of submodules of M.

Put $K = \bigcup_{i=1}^{\infty} K_i$ → K is submodule of M.

→ K is a finitely generated submodule of M

→ $K = Rk_1 + Rk_2 + \ldots + Rk_n$

→ each K_j is in K_i's

→ $\exists$ m such that $k_1, k_2, \ldots, k_r \in K_m \quad \forall\ n \geq m$

∴ M is Noetherian module.

Examples.

1. The $\mathbb{Z}$- module $\mathbb{Z}$ is Noetherian module (every submodule of the $\mathbb{Z}$-module $\mathbb{Z}$ (= $n\mathbb{Z}$ cyclic) is finitely generated) which is not Artinian ($2\mathbb{Z} > 4\mathbb{Z} > 8\mathbb{Z} > \ldots > 2^n\mathbb{Z} > \ldots$. is a chain of ideals of Z that does not terminate)
2. The ring of integers $\mathbb{Z}$ is Noetherian (every principal ideal ring is Noetherian).

3. Q is not Noetherian module (since the $\mathbb{Z}$- module Q is not finitely generated).
4. A division ring D is Artinian and Noetherian since the only right or left ideals of D are 0 and D.
5. Every finite module is an Artinian module.

Remark. Every nonzero Artinian module contains a simple submodule.
Proof. let $0 \neq M$ be an Artinian module.
If M is a simple module, then we are done.
If not, $\exists\ 0 \neq M_1$ submodule of M. If M_1 is a simple, then we are done.
If not, $\exists\ 0 \neq M_2$ submodule of M_1. If M_2 is a simple, then we are done.
If not, $\exists\ 0 \neq M_3$ submodule of M_2. If M_3 is a simple, then we are done.
So there is a descending chain

$$M \geq M_1 \geq M_2 \geq M_3 \geq \ldots$$

of submodules of M. Since M is an Artinian module, then the family $\{M_i\}_{i \in I}$ of the chain has minimal element and this element is the simple submodule.

Proposition. Let $0 \rightarrow N \xrightarrow{i} M \xrightarrow{\pi} \frac{M}{N} \rightarrow 0$be a short exact sequence of R-modules and module homomorphism. Then M is Noetherian (resp. Artinian) iff both N (Artinian) and $\frac{M}{N}$ are Noetherian (Artinian) (resp. Artinian).

Proof. →) Suppose that M is a Noetherian module and N submodule of M. so every submodule of N is a submodule of M. so N is Noetherian.
Let

$$\frac{M_1}{N} \leq \frac{M_2}{N} \leq \frac{M_3}{N} \leq \ldots$$

be an ascending chain of submodules of $\frac{M}{N}$, where

$$M_1 \leq M_2 \leq M_3 \leq \ldots$$

is an ascending chain of submodules of M which contain N. But M Noetherian, $\exists$m such that $M_n = M_m$ for all $n \geq m$.

$$\therefore \frac{M}{N} \text{ is Noetherian module.}$$

$\leftarrow$) Suppose that N and $\frac{M}{N}$ are Noetherian modules. Let

$$M_1 \leq M_2 \leq M_3 \leq \ldots$$

be an ascending chain of submodules of M. Then

$$M_1 \cap N \leq M_2 \cap N \leq M_3 \cap N \leq \ldots$$

is an ascending chain of submodules of N, so there is an integer $m_1 \geq 1$ such that $M_n \cap N = M_{m_1} \cap N$ for all $n \geq m_1$. Also,

$$\frac{M_1+N}{N} \leq \frac{M_2+N}{N} \leq \frac{M_3+N}{N} \leq \ldots$$

is an ascending chain of submodules of $\frac{M}{N}$ and there is an integer $m_2 \geq 1$ such that $\frac{M_n+N}{N} = \frac{M_{m_2}+N}{N}$ for all $n \geq m_2$. Let m = max.$\{m_1, m_2\}$. Then for all $n \geq m$,

$$M_n \cap N = M_m \cap N \text{ and } \frac{M_n+N}{N} = \frac{M_m+N}{N}$$

If $n \geq m$ and $x \in M_n$, then $x + N \in \frac{M_n+N}{N} = \frac{M_m+N}{N}$, so there is a $y \in M_m$ such that $x + N = y + N$ implies that $x - y \in N$ and since $M_m \leq M_n$ we have $x - y \in M_n \cap N = M_m \cap N$ when $n \geq m$ If $x - y = z \in M_m \cap N$, then $x = y + z \in M_m$, so $M_n \leq M_m$. Hence, $M_n = M_m$ whenever $n \geq m$, so M is Noetherian.

Remark. In general, if the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact, then B is Noetherian (Artinian) if and only if each of A and C is Noetherian (Artinian).

Example. Let M_1 and M_2 be R-modules. Then $M_1 \oplus M_2$ is Noetherian (Artinian) iff each of M_1 and M_2 is Noetherian (Artinian). (i.e every finite direct sum of Noetherian (Artinian)is Noetherian (Artinian)
(The proof is done using the short exact sequence

$$0 \to M_1 \xrightarrow{J_1} M_1 \oplus M_2 \xrightarrow{\rho_2} M_2 \to 0)$$

Theorem. Let $\alpha : M \to \acute{M}$ be an epimorphism. If M is Noetherian (Artinian), then so is $\acute{M}$.

Proof. Since kerα is a submodule of M, then the sequence

$$0 \to ker\alpha \xrightarrow{i} M \xrightarrow{\pi} \frac{M}{ker\,\alpha} \to 0$$

is a short exact sequence. By hypothesis, M is Noetherian, implies that $\frac{M}{ker\,\alpha}$ is Noetherian. But $\frac{M}{ker\,\alpha} \approx \acute{M}$(first isomorphism theorem) and $\frac{M}{ker\,\alpha}$ is Noetherian, so $\acute{M}$ is a Noetherian.

Theorem. The following are equivalent for a ring R.

1. R is right Noetherian.
2. Every finitely generated R-module is Noetherian.

Proof.($1\rightarrow 2$) let M be a finite generated over a Noetherian ring R.
$\exists$ $x_1, x_2, \ldots, x_n \in M$ such that $M = Rx_1 + Rx_2 + \ldots + Rx_n$. since R is Noetherian, then so is the finite direct sum of copies of R. Define
$\alpha : R^{(n)} \rightarrow M$ by : $\alpha(r_1, r_2, \ldots, r_n) = r_nx_1 + r_nx_2 + \ldots + r_nx_n$.
It's clear that α is a well-define, homomorphism and onto. So, Imα = M is Noetherian.
($2 \rightarrow 1$)Since R = <1>, so R is finitely generated and hence R is Noetherian.

References

1. P.E. Bland, "Rings and Their modules", New York, 2011.
2. T.W. Hungerford, "Algebra", New York, 2000.
3. D.M. Burton, "Abstract and linear algebra", London, 1972.
4. M.F. Atiyah, "Introduction to Commutative Algebra", University of Oxford, 1969.